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LETTER TO THE EDITOR

Induced module construction for highest-weight representations of $U_q(\mathrm{gl}(n))$ at roots of unity

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Abstract. A method is investigated for inducing highest-weight representations for the quantum group $U_q(\mathrm{gl}(n))$ from the canonical subalgebra $U_q(\mathrm{gl}(n-1))$ when q is a root of unity. We classify the irreps into two types, typical and atypical, where the former is a generalization of the class of irreps with maximal dimensionality. The structures of both the typical and atypical irreps are studied; in particular, a sufficiency condition is given for an irrep to be typical. As examples, we consider flat representations induced from a one-dimensional representation of the canonical subalgebra and representations induced from vector representation.

Quantum groups [1, 2] are a special class of Hopf algebras which have attracted considerable attention because of their application in a variety of areas of mathematical physics. At generic values of the deformation parameter q, quantum groups are quasitriangular Hopf algebras [1], admitting a universal R-matrix which plays an important role in solving the Yang–Baxter equation.

When q is a root of unity, the structure and representation theory of quantum groups alters dramatically due to the centre becoming augmented by additional elements. Considerable research has been undertaken to study these algebras and their representations using various techniques including Gelfand–Tsetlin bases, auxiliary algebras and q-boson calculus [3–8]. Although these algebras are not quasitriangular in the usual sense, they possess a more general property called autoquasitriangularity as defined by Reshetikhin [9]. Aside from their mathematical interest, quantum groups at roots of unity have applications in a range of areas. It is well known that the chiral Potts model and its generalizations [10, 11] are based on representations of $U_q((n))$ when q is a root of unity. Other examples include field theories [12] and the construction of three manifold invariants [13, 14].

In this letter, we wish to investigate a method of inducing highest-weight representations of the quantum group $U_q(\mathrm{gl}(n))$, when $q^N = 1$, from a representation of the canonical subalgebra $U_q(\mathrm{gl}(n-1))$. These representations do not necessarily admit a lowest weight and their dimensions are bounded by $N^{\frac{1}{2}n(n-1)}$, features which are not shared by representations at generic q. Our approach is inspired by the work of Kac [15] on the induced module construction for finite-dimensional highest-weight representations of basic classical Lie superalgebras and its generalizations [16, 17]. It is also closely related to the Biedenharn–Lohe construction [18] of $U_q(\mathrm{gl}(n))$ irreps at generic q based on the Borel–Weil theory. Many techniques developed in [18] may be adopted into our framework to investigate the representations at roots of unity in a detailed fashion.

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Although our construction does not yield the most general type of representations at roots of unity, it is quite simple to apply and, in principle, allows one to investigate the highest-weight representations systematically. The construction also naturally classifies the irreps into two types, typical and atypical, where the former is a significant generalization of the class of irreps with maximal dimensionality studied by De Concini and Kac [3]. We will study the structures of both types of irreps to some extent; in particular, we give a sufficient condition for an irrep to be typical. As we will see, the structures of the typical irreps can be understood relatively easily, while those of the atypicals are harder to study. As examples, flat representations and those induced from the vector irrep of $U_q(\mathrm{gl}(n))$ are considered in detail.

The quantum group $U_q(\mathrm{gl}(n))$ is a unital algebra generated by the elements $E^a_{a\pm1}$, $q^{\pm E^a_a}$, $a, a\pm 1 = 1, 2, \ldots, n$, where $0 \neq q \in \mathbb{C}$ is an indeterminate, subject to the following constraints:

$$q^{\pm E^a_a} q^{\pm E^b_b} = q^{\pm E^b_b} q^{\pm E^a_a} \qquad q^{E^a_a} q^{-E^a_a} = I$$

$$q^{E^a_a} E^b_{b\pm1} q^{-E^a_a} = q^{(\delta^b_a - \delta^a_{b\pm1})} E^b_{b\pm1}$$

$$[E^a_{a+1}, E^{b+1}_b] = \delta^a_b \frac{(q^{h_a} - q^{-h_a})}{(q - q^{-1})}$$

where

$$q^{h_a} = q^{E^a_a - E^{a+1}_{a+1}}$$

$$E^a_{a+1} E^b_{b+1} = E^b_{b+1} E^a_{a+1} \qquad |a-b| \geqslant 2$$

$$E^{a+1}_a E^{b+1}_b = E^{b+1}_b E^{a+1}_a \qquad |a-b| \geqslant 2$$

$$(E^a_{a+1})^2 E^{a\pm1}_{a\pm1+1} - (q + q^{-1}) E^a_{a+1} E^{a\pm1}_{a\pm1+1} E^a_{a+1} + E^{a\pm1}_{a\pm1+1} (E^a_{a+1})^2 = 0$$

$$(E^{a+1}_a)^2 E^{a\pm1+1}_{a\pm1} - (q + q^{-1}) E^{a+1}_a E^{a\pm1+1}_{a\pm1} E^{a+1}_a + E^{a\pm1+1}_{a\pm1} (E^{a+1}_a)^2 = 0$$

and where [,] denotes the usual commutator.

$U_q(\mathrm{gl}(n))$ admits a Hopf algebra structure, with co-product $\Delta : U_q(\mathrm{gl}(n)) \to U_q(\mathrm{gl}(n)) \otimes U_q(\mathrm{gl}(n))$, co-unit $\epsilon : U_q(\mathrm{gl}(n)) \to \mathbb{C}$ and antipode $S : U_q(\mathrm{gl}(n)) \to U_q(\mathrm{gl}(n))$. However, we omit the details of these mappings as they will not be required for the purpose of this letter.

We can construct root vectors of $U_q(\mathrm{gl}(n))$ in the following way.

$$E^a_b = E^a_c E^c_b - q^{-1} E^c_b E^a_c$$

$$E^b_a = E^b_c E^c_a - q E^c_a E^b_c \qquad a < c < b. \tag{1}$$

It can be verified that such vectors satisfy the following relations $\forall a < c < b$:

$$E^b_a E^b_c = q E^b_c E^b_a \qquad E^a_b E^c_b = q E^c_b E^a_b$$

$$E^c_a E^b_a = q E^b_a E^c_a \qquad E^a_c E^a_b = q E^a_b E^a_c$$

$$[E^a_c, (E^b_a)^p] = q[-p] E^b_c (E^b_a)^{p-1} q^{E^a_a - E^c_c}$$

$$E^c_a(E^b_c)^p = q^{-p}(E^b_c)^p E^c_a + q^{-1}[-p](E^b_c)^{p-1}E^b_a$$

$$[E^a_b, (E^b_c)^p] = q^{-p}[p](E^b_c)^{p-1}E^a_c q^{E^c_c - E^b_b}$$

$$[E^c_b, (E^b_a)^p] = [p]E^c_a(E^b_a)^{p-1}q^{E^b_b - E^c_c}$$

$$[E^a_b, (E^b_a)^p] = [p](E^b_a)^{p-1}\left(\frac{q^{E^a_a - E^b_b + 1 - p} - q^{E^b_b - E^a_a + p - 1}}{q - q^{-1}}\right)$$

$$[(E^a_b)^p, (E^b_a)^p] = \sum_{j=1}^{p}\begin{bmatrix} p \\ j \end{bmatrix}[j]!(E^b_a)^{p-j}\prod_{r=j+1-2p}^{2j-2p}\left(\frac{q^{E^a_a - E^b_b + r} - q^{E^b_b - E^a_a - r}}{q - q^{-1}}\right)(E^a_b)^{p-j}$$

where

$$[j] = \left(\frac{q^j - q^{-j}}{q - q^{-1}}\right) \qquad [j]! = [j].[j-1]! \qquad [1]! = 1 \qquad \begin{bmatrix} p \\ j \end{bmatrix} = \frac{[p]!}{[p-j]![j]!}. \quad (2)$$

Hereafter, we let $q \in \mathbb{C}$ be a fixed primitive root of unity such that $q^N = 1$. It is known [3] that in such a case the elements

$$(E^a_b)^N \qquad \forall 1 \leqslant a, b \leqslant n \qquad a \neq b$$

are central. It is worth mentioning that when N is even, the elements

$$(E^a_b)^{\frac{1}{2}N}$$

are central with respect to algebra $U_q((n))$ [3]. However, we will restrict our attention to $U_q(\mathrm{gl}(n))$ and, in what follows, we will be concerned with inducing highest-weight representations of $U_q(\mathrm{gl}(n))$ from representations of the canonical subalgebra $U_q(\mathrm{gl}(n-1))$. Throughout, we will denote the eigenvalues of the above central elements by

$$\chi((E^a_b)^N) = \alpha_{ab} \qquad \alpha_{ab} \in \mathbb{C}.$$

Since we are considering highest-weight representations, we then necessarily have

$$\alpha_{ab} = 0 \qquad \forall 1 \leqslant a < b \leqslant n.$$

Next we will investigate a procedure for constructing highest-weight representations of $U_q(\mathrm{gl}(n))$ induced from representations of $U_q(\mathrm{gl}(n-1))$. Such a procedure may be used inductively along the subalgebra chain

$$U_q(\mathrm{gl}(2)) \subset U_q(\mathrm{gl}(3)) \subset \cdots \subset U_q(\mathrm{gl}(n-1)) \subset U_q(\mathrm{gl}(n)).$$

Let $V_0(\Lambda^{(n-1)})$, $\Lambda = \sum_{a=1}^{n-1}\Lambda_a\varepsilon_a$ denote an irreducible $U_q(\mathrm{gl}(n-1))$ module with basis v_i and a highest-weight vector v_0 which satisfies

$$E^a_b v_0 = 0 \qquad \forall 1 \leqslant a < b \leqslant n-1$$

$$q^{E^a_a}v_0 = q^{\Lambda_a}v_0 \qquad \forall 1 \leqslant a \leqslant n-1.$$

We also let $\overline{U_q(\mathrm{gl}(n))}$ denote the algebra generated by $U_q(\mathrm{gl}(n-1)) \cup \{E^n_{n-1}, q^{E^n_n}\}$. We construct the following elements of $\overline{U_q(\mathrm{gl}(n))}$

$$\Gamma(p) = (E^n_1)^{p_1}(E^n_2)^{p_2} \ldots (E^n_{n-1})^{p_{n-1}}$$

where $p = \sum_{a=1}^{n-1} p_a \varepsilon_a$, and construct the following basis vectors:

$$\Gamma_j(p) = \Gamma(p) \otimes v_j.$$

The above vectors generate a $\overline{U_q(\mathrm{gl}(n))}$ module with action given by

$$\begin{aligned}
&E^a_b \Gamma_j(p) = \Gamma(p) \otimes E^a_b v_j + q^{2+p_b-\sum_{l=a}^{b-1} p_l}[-p_a]\Gamma(p - \varepsilon_a + \varepsilon_b) \otimes q^{E^a_a - E^b_b} v_j \\
&E^b_a \Gamma_j(p) = q^{p_a - p_b}\Gamma(p) \otimes E^b_a v_j + q^{p_a - \sum_{l=a+1}^{b} p_l}[p_a]\Gamma_j(p + \varepsilon_a - \varepsilon_b) \\
&E^n_a \Gamma_j(p) = q^{-\sum_{l=1}^{a-1} p_l}\Gamma_j(p + \varepsilon_a) \\
&q^{E^a_a}\Gamma_j(p) = q^{-p_a}\Gamma(p) \otimes q^{E^a_a} v_j \\
&q^{E^n_n}\Gamma_j(p) = q^{\Lambda_n + \sum_{l=1}^{n-1} p_l}\Gamma_j(p) \qquad \forall 1 \leqslant a < b \leqslant n-1
\end{aligned} \tag{3}$$

where $\Lambda = \Lambda_{n-1} + \Lambda_n \varepsilon_n$ and $\Lambda_n \in \mathbb{C}$ is arbitrary. We denote this module by $\overline{V(\Lambda)}$.

Observe that we may write

$$\overline{V(\Lambda)} = \bigoplus_{i=0}^{N-1} T_i \tag{4}$$

where

$$T_i = \left\{\Gamma_j(p) : \sum_{l=0}^{n-1} p_l = i \bmod N\right\}. \tag{5}$$

Each of the T_i gives rise to a $U_q(\mathrm{gl}(n-1))$ module and $q^{E^n_n}$ acts as a level operator in the sense that

$$q^{E^n_n} T_i = q^{\Lambda_n + i} T_i.$$

In order to make $\overline{V(\Lambda)}$ a $U_q(\mathrm{gl}(n))$ module, we need to define the action of E^{n-1}_n. Let us define

$$E^{n-1}_n(I \otimes V(\Lambda^{(n-1)})) = 0$$

which can be shown to imply

$$E^a_n(I \otimes V(\Lambda^{(n-1)})) = 0 \qquad \forall 1 \leqslant a \leqslant n-1.$$

Using (2) and (3), we can then write

$$
\begin{aligned}
E_n^{n-1}\Gamma_j(p) = {} & [p_{n-1}]\Gamma(p-\varepsilon_{n-1}) \otimes \left(\frac{q^{E_{n-1}^{n-1}-\Lambda_n+1-p_{n-1}} - q^{\Lambda_n - E_{n-1}^{n-1}+p_{n-1}-1}}{q-q^{-1}} \right) v_j \\
& + \sum_{a=1}^{n-2} q^{\Lambda_n - 1 + p_a + p_{n-1}} [p_a] \\
& \times \left[q^{\sum_{l=a+1}^{n-2} p_l} \Gamma(p-\varepsilon_a) \otimes E_a^{n-1} q^{-E_{n-1}^{n-1}} v_j + [-p_{n-1}]\Gamma(p-\varepsilon_{n-1}) \otimes q^{-E_{n-1}^{n-1}} v_j \right]
\end{aligned} \tag{6}
$$

thus providing a representation of $U_q(\mathrm{gl}(n))$ which possesses a highest-weight vector $I \otimes v_0$ (not necessarily unique). It should be apparent that highest weights satisfying

$$\Lambda_a = \nu_a + k_a N \qquad k_a \in \mathbb{Z} \qquad \forall 1 \leqslant a \leqslant n$$

give rise to equivalent modules $\overline{V(\Lambda)}$ and $\overline{V(\nu)}$.

Note also that $\overline{V(\Lambda)}$ is not necessarily an irreducible module. Let M denote the unique maximal proper submodule of $\overline{V(\Lambda)}$. We set

$$V(\Lambda) = \overline{V(\Lambda)}/M.$$

Then $V(\Lambda)$ is irreducible. If $M \neq 0$, we say that $V(\Lambda)$ is *atypical*. On the other hand, if $\overline{V(\Lambda)}$ is irreducible, we set $V(\Lambda) = \overline{V(\Lambda)}$ and we refer to it as being *typical*. As we will see presently, the structures of the typical irreps can be understood rather easily, while those of the atypicals are harder to study.

Let us consider the case when

$$\alpha_{na} = 0 \qquad \forall 1 \leqslant a < n \tag{7}$$

and derive necessary and sufficient conditions for $V(\Lambda)$ to be typical. It is instructive to point out that when $V_0(\Lambda^{(n-1)})$ possesses a lowest-weight state then the induced module $\overline{V(\Lambda)}$ also admits a lowest-weight state under condition (7).

We begin by defining (cf (4) and (5))

$$\left\{ T_i = \Gamma_j(p) : \sum_{l=1}^{n-1} p_l = i \right\} \tag{8}$$

and observing that

$$\overline{V(\Lambda)} = \bigoplus_{i=0}^{K} T_i \tag{9}$$

where each of the T_i constitutes a $U_q(\mathrm{gl}(n-1))$ module and $K = (n-1)(N-1)$. That is to say, we can further split the $U_q(\mathrm{gl}(n-1))$ levels of equation (4) under condition (7). In particular, we can identify a lowest level, namely T_K. We have the following result.

Lemma 1. Any submodule of $\overline{V(\Lambda)}$ contains the $U_q(\mathrm{gl}(n-1))$ module T_K.

Proof. Given any vector contained in a submodule of $\overline{V(\Lambda)}$, we can obtain a vector in T_K by repeated application of the generators E^n_a, $1 \leqslant a < n$. We note that

$$\Gamma(N-1) \equiv \{\Gamma(p) : p_a = N-1 \quad \forall 1 \leqslant a < n\}$$

commutes with the $U_q(\mathrm{gl}(n-1))$ generators. Hence, we can generate the whole of T_K. □

Equipped with lemma 1, we define

$$Y_a = (E^{n-1}_n)^{N-1} \cdots (E^a_n)^{N-1} (E^n_a)^{N-1} \cdots (E^n_{n-1})^{N-1} \otimes v_0.$$

It is apparent that module $\overline{V(\Lambda)}$ is irreducible if, and only if, $Y_1 \neq 0$. By repeated application of equations (2), we have

$$\begin{aligned} Y_a &= (E^{n-1}_n)^{N-1} \cdots (E^{a+1}_n)^{N-1} [N-1]! \prod_{r=0}^{N-2} \left(\frac{q^{E^a_a - E^n_n - r} - q^{E^n_n - E^a_a + r}}{q - q^{-1}} \right) \\ &\quad \times (E^n_{a+1})^{N-1} \cdots (E^n_{n-1})^{N-1} \otimes v_0 \\ &= [N-1]! \prod_{r=0}^{N-2} \left(\frac{q^{\lambda_a - \lambda_n - r + n - a - 1} - q^{\lambda_n - \lambda_a + r - n + a + 1}}{q - q^{-1}} \right) Y_{a+1} \\ &= [N-1]! \prod_{r=1}^{N-1} \left(\frac{q^{(\lambda+\rho, \varepsilon_a - \varepsilon_n) - r} - q^{-(\lambda+\rho, \varepsilon_a - \varepsilon_n) + r}}{q - q^{-1}} \right) Y_{a+1} \end{aligned}$$

where $\rho = \frac{1}{2} \sum_{a=1}^n (n+1-2a)\varepsilon_a$ is the half sum of positive roots of $\mathrm{gl}(n)$. The above result leads to the following proposition (cf [3]).

Proposition 1. The module $V(\Lambda)$ is typical (i.e. $\overline{V(\Lambda)}$ is irreducible) if, and only if,

$$(\Lambda + \rho, \varepsilon_a - \varepsilon_n) \notin \mathbb{Z}/N\mathbb{Z} \qquad \forall 1 \leqslant a < n.$$

In view of proposition 1, it is apparent that for generic values of parameter Λ_n, the induced module construction yields a one-parameter family of typical modules. It is straightforward to show that a typical module $V(\Lambda)$ induced from the $U_q(\mathrm{gl}(n-1))$ module $V_0(\Lambda^{(n-1)})$ has

$$\dim V(\Lambda) = N^{(n-1)} \dim V_0(\Lambda^{(n-1)})$$

and also a vanishing q-dimension [19]. Also, the typical irreps are a significant extension of the class of irreps with maximal dimensionality studied in [3]. Proceeding down the chain of subalgebras, it is clear that a module $V(\Lambda)$ will be of maximal dimension if, and only if,

$$(\Lambda + \rho,\ \varepsilon_i - \varepsilon_j) \notin \mathbb{Z}/N\mathbb{Z} \qquad \forall i, j$$

which is exactly the criterion given in [3]. It is also worthwhile to point out that if $V(\Lambda)$ is typical under condition (7), then it is also typical for general values of α_{na}, although the converse is not true.

To demonstrate how our construction works, we apply it to obtain the irreducible representations of $U_q(\mathrm{gl}(n))$ induced from the trivial and vector representations of $U_q(\mathrm{gl}(n-$

1)). The irreducible representations induced from the trivial $U_q(\mathrm{gl}(n-1))$ module are termed *flat* (i.e. the weight spectrum is multiplicity free) and have been considered in [5] using Gelfand–Tsetlin basis states. Here we will reconsider these representations using our alternative approach, which, as we will see, is very simple.

Let v_0 provide a one-dimensional module of $U_q(\mathrm{gl}(n-1))$ with the usual (trivial) action. We construct the $U_q(\mathrm{gl}(n))$ module $\overline{V(\Lambda_n\varepsilon_n)}$ with basis vectors

$$\Gamma_0(p) \otimes v_0.$$

From equations (3) and (6), the action of the $U_q(\mathrm{gl}(n))$ generators is given by

$$\begin{aligned}
E^a_b\Gamma_0(p) &= q^{2+p_b-\sum_{l=a}^{b-1} p_l}[-p_a]\Gamma_0(p-\varepsilon_a+\varepsilon_b)\\
E^b_a\Gamma_0(p) &= q^{p_a-\sum_{l=a+1}^{b} p_l}[p_a]\Gamma_0(p+\varepsilon_a-\varepsilon_b)\\
E^n_a\Gamma_0(p) &= q^{-\sum_{l=1}^{a-1} p_l}\Gamma_0(p+\varepsilon_a)\\
q^{E^a_a}\Gamma_0(p) &= q^{-p_a}\Gamma_0(p)\\
q^{E^n_n}\Gamma_0(p) &= q^{\Lambda_n+\sum_{l=1}^{n-1} p_l}\Gamma_0(p)\\
E^{n-1}_n\Gamma_0(p) &= \left\{[1-p_{n-1}-\Lambda_n][p_{n-1}] + \sum_{a=1}^{n-2} q^{\Lambda_n-1+p_a+p_{n-1}}[p_a][-p_{n-1}]\right\}\Gamma_0(p-\varepsilon_{n-1})\\
&\forall 1 \leqslant a < b \leqslant n.
\end{aligned} \tag{10}$$

The explicit action of the generators E^a_n, $1 \leqslant a < n$, is obtained from (1) and (10).

We now consider the conditions under which $\overline{V(\Lambda)}$ is irreducible. First, if any of the α_{na}, $1 \leqslant a < n$ are non-zero then $\overline{V(\Lambda)}$ is necessarily irreducible. To see this, given any vector $\Gamma_0(p)$, application of generators E^b_a, $b \neq a$ an appropriate number of times yields

$$(E^n_a)^l \otimes v_0$$

for some l. $\alpha_{na} \neq 0$ immediately implies that $V(\Lambda)$ is typical. On the other hand, if $\alpha_{na} = 0$ $\forall 1 \leqslant a < n$, then proposition 1 implies that $\overline{V(\Lambda_n\varepsilon_n)}$ is irreducible if, and only if, $\Lambda_n \notin \mathbb{Z}$. We remark that for $\Lambda_n \in \mathbb{Z}$ in the range $0 \geqslant \Lambda_n > N$, the irreducible module $V(\Lambda_n\varepsilon_n)$ is simply the dual of the rank $-\Lambda_n$ symmetric representation which has dimension

$$\frac{(n-1-\Lambda_n)!}{(n-1)!(-\Lambda_n)!}.$$

This leads to the following classification.

Proposition 2. All of the irreducible modules $V(\Lambda_n\varepsilon_n)$ of $U_q(\mathrm{gl}(n))$ have dimension $(n-1)^{(N-1)}$ unless $\alpha_{na} = 0$, $\forall 1 \leqslant a < n$ and $\Lambda_n \in \mathbb{Z}$. In such a case, $V(\Lambda_n\varepsilon_n)$ has dimension

$$\frac{(n-1+N-\Lambda_n)!}{(n-1)!(N-\Lambda_n)!}$$

where $\Lambda_n \in \mathbb{R}$ is chosen in the range $0 < \Lambda_n \leqslant N$.

Next we construct representations induced from the vector module of $U_q(\mathrm{gl}(n-1))$ with basis $\{v_j\}_{j=1}^{n-1}$. We change our convention here for ease of notation and let v_1 denote the highest-weight state. The action of the $U_q(\mathrm{gl}(n-1))$ generators on this module is particularly simple, viz

$$E_b^a v_j = \delta_{bj} v_a \qquad \forall a \neq b$$

$$q^{E_a^a} v_j = q^{\delta_{aj}} v_j \qquad \forall 1 \leqslant a \leqslant n-1.$$

Equations (3) and (6) then take the form

$$E_b^a \Gamma_j(p) = \delta_{bj} \Gamma_a(p) + q^{2+p_b-\sum_{l=a}^{b-1} \delta_{aj} - \delta_{bj}} [-p_a] \Gamma_j(p - \varepsilon_a + \varepsilon_b)$$

$$E_a^b \Gamma_j(p) = \delta_{aj} q^{p_a - p_b} \Gamma_b(p) + q^{p_a - \sum_{l=a+1}^{b} p_l} [p_a] \Gamma_j(p + \varepsilon_a - \varepsilon_b)$$

$$E_a^n \Gamma_j(p) = q^{-\sum_{l=1}^{a-1} p_l} \Gamma_j(p + \varepsilon_a)$$

$$q^{E_a^a} \Gamma_j(p) = q^{-p_a + \delta_{aj}} \Gamma_j(p)$$

$$q^{E_n^n} \Gamma_j(p) = q^{\Lambda_n + \sum_{l=1}^{n-1} p_l} \Gamma_j(p)$$

$$E_n^{n-1} \Gamma_j(p) = [p_{n-1}][1 - p_{n-1} - \Lambda_n - \delta_{j(n-1)}] \Gamma_j(p - \varepsilon_{n-1}) + \sum_{a=1}^{n-2} q^{\Lambda_n - 1 + p_a + p_{n-1}} [p_a]$$
$$\times \left[\delta_{aj} q^{\sum_{l=a+1}^{n-2} p_l} \Gamma_{n-1}(p - \varepsilon_a) + q^{-\delta_{j(n-1)}} [-p_{n-1}] \Gamma_j(p - \varepsilon_{n-1}) \right].$$

Under condition (7), we deduce from proposition 1 that the above induced module with highest weight $\varepsilon_1 + \Lambda_n \varepsilon_n$ is irreducible provided $\Lambda_n \notin \mathbb{Z}$. Hence, for generic values of Λ_n, the above construction yields a one-parameter family of irreducible modules with dimension $(n-1)N^{(n-1)}$. For general values of α_{na}, $V(\Lambda)$ is also typical provided $\Lambda_n \notin \mathbb{Z}$. The treatment of the atypical modules, which we will not consider here, is comparatively more complicated.

In conclusion, in this letter, we have developed a method of constructing finite-dimensional highest-weight representations of $U_q(\mathrm{gl}(n))$ when $q^N = 1$. These representations do not, in general, admit a lowest-weight vector and, therefore, are not simply a deformation of the $q = 1$ case. Indeed, there are no lexicality conditions imposed on the highest weight unlike the $q = 1$ case and, as is known [3], the dimensions of the irreps are bounded. We have classified the irreps into two types, typical and atypical. The former represents a significant extension of the irreps with maximal dimensionality studied in [3]. Also, the structures of the typical irreps are rather well understood.

As examples, we constructed flat representations of $U_q(\mathrm{gl}(n))$ induced from a trivial representation of $U_q(\mathrm{gl}(n-1))$ and representations induced from the vector representation. Finally, we would like to mention that a similar method has been applied to construct highest-weight representations of the quantum supergroup $U_q(\mathrm{gl}(m|n))$ [17].

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